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The Supremum Norm of Reciprocals of Christoffel Functions for Erdős Weights

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Let $W := e^{-Q}$, where Q is even and of faster than polynomial growth at infinity. For sufficiently smooth Q, define the function $T(x) := 1 + xQ''(x)/Q'(x)$, which has limit ∞ at ∞. Let a_n denote the nth Mhaskar–Rahmanov–Saff number for W, and let $\lambda_n(W^2, x)$ denote the nth Christoffel function. We show that as $n \to \infty$,

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \sim \frac{n}{a_n} T(a_n)^{1/2}$$

and

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x)\, |1-(x/a_n)^2|^{1/2} \sim \frac{n}{a_n}.$$

1. INTRODUCTION AND STATEMENT OF RESULTS

Let W: $\mathbb{R} \to \mathbb{R}$ be even, positive, continuous, and such that all power moments

$$\int_{-\infty}^{\infty} x^j W(x)\, dx, \qquad j = 0, 1, 2, ...$$

exist. Associated with W^2 are the *orthonormal polynomials* p_j of degree j, $j = 0, 1, 2, ...$ satisfying

$$\int_{-\infty}^{\infty} p_j(x)\, p_k(x)\, W^2(x)\, dx = \delta_{jk}, \qquad j, k = 0, 1, 2, ... \tag{1.1}$$

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The nth *Christoffel function* is

$$\lambda_n(W^2, x) := 1 \Big/ \sum_{j=0}^{n-1} p_j^2(x), \tag{1.2}$$

$x \in \mathbb{R}$, $n = 1, 2, 3, \ldots$.

In many questions concerning the convergence and the approximation properties of orthonormal expansions associated with W^2, the Christoffel functions play a crucial role [4, 14]. In particular, the rate of growth as $n \to \infty$ of

$$\lambda_n^{\max} := \max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x)$$

$$= \max_{x \in \mathbb{R}} \sum_{j=0}^{n-1} p_j^2(x)\, W^2(x) \tag{1.3}$$

was determined by Freud [4] for a class of weights containing $\exp(-|x|^\alpha)$, $\alpha \geqslant 2$ and applied to prove $(C, 1)$ boundedness of the partial sums of orthonormal expansions associated with W^2. Other authors [5, 6, 7, 15] considered weights including $\exp(-|x|^\alpha)$, $0 < \alpha < 2$.

For the weights $\exp(-|x|^\alpha)$, $\alpha > 0$, results in [3–7, 15] imply that

$$\lambda_n^{\max} \sim \begin{cases} n^{1-1/\alpha}, & \alpha > 1, \\ \log n, & \alpha = 1, \\ 1, & \alpha < 1. \end{cases} \tag{1.4}$$

Here $\sim$ means that the ratio of the quantities on either side is bounded above and below by positive constants independent of n. To further elucidate this type of result, we need the *Mhaskar–Rahmanov–Saff* number a_n [11, 12].

Let $W := e^{-Q}$, where Q is even and differentiable in $(0, \infty)$, while $xQ'(x)$ is strictly increasing in $(0, \infty)$, with limits 0 and ∞ at $x = 0$ and ∞, respectively. Then a_u is the root of

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}, \qquad u > 0. \tag{1.5}$$

In general a_u grows roughly like the inverse function of $uQ'(u)$ as $u \to \infty$. If $Q^{[-1]}$ denotes the inverse of Q, and if Q is of at least polynomial growth at infinity, then a_u grows roughly like $Q^{[-1]}(u)$ as $u \to \infty$. For classes of weights $W^2 = e^{-2Q}$, where $Q(x)$ grows faster than $|x|^\alpha$, some $\alpha > 1$, the results of [4, 6, 7] imply that

$$\lambda_n^{\max} \sim n/a_n, \qquad n \to \infty. \tag{1.6}$$

Pointwise asymptotics for $\lambda_n(W^2, x)$ appear in [1, 9, 13, 16].

The quoted results treat the Q that is of polynomial growth at infinity. What happens when Q is faster than polynomial growth at infinity? In contrast to (1.6), it turns out that

$$\lim_{n \to \infty} \lambda_n^{\max}/(n/a_n) = \infty. \tag{1.7}$$

To precisely state our result, which follows (in a non-trivial manner) from pointwise asymptotics for $\lambda_n(W^2, x)$ in [9], we need:

DEFINITION 1.1. Let $W := e^{-Q}$, where Q is even and continuous in $\mathbb{R}$; Q''' exists in $(0, \infty)$, and Q' is positive in $(0, \infty)$. Let

$$T(x) := 1 + xQ''(x)/Q'(x), \qquad x \in (0, \infty), \tag{1.8}$$

be increasing in $(0, \infty)$, with

$$\lim_{x \to 0+} T(x) = T(0+) > 1, \tag{1.9}$$

$$\lim_{x \to \infty} T(x) = \infty, \tag{1.10}$$

and for each $\varepsilon > 0$,

$$T(x) = O(Q'(x)^{\varepsilon}), \qquad x \to \infty. \tag{1.11}$$

Assume further that

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)}, \qquad x \text{ large enough}, \tag{1.12}$$

and for some $C > 0$,

$$\frac{|Q'''(x)|}{Q'(x)} \leqslant C \left\{ \frac{Q'(x)}{Q(x)} \right\}^2, \qquad x \text{ large enough}. \tag{1.13}$$

Then we say that W is an *Erdős weight of class* 3 and write $W \in SE^*(3)$.

Remarks. (a) The limit (1.10) implies that $Q(x)$ grows faster than any polynomial, while (1.11) is a weak regularity condition: one typically has [8, 9]

$$T(x) = O([\log Q'(x)]^{1+\varepsilon}), \qquad x \to \infty, \tag{1.14}$$

for each $\varepsilon > 0$. The restriction (1.9) simplifies analysis, but can be weakened.

(b) The class $SE^*(3)$ is contained in the class $SE(3)$ of [9], for in [9] we take only $\varepsilon = \frac{1}{15}$ in (1.11).

(c) As examples of $W \in SE^*(3)$, we mention

$$W(x) := \exp(-\exp_k(|x|^\alpha)), \qquad x \in \mathbb{R}, \tag{1.15}$$

$\alpha > 1$, $k \geqslant 1$, where $\exp_k$ denotes the kth iterated exponential $\exp(\exp \cdots)$. Another example is

$$W(x) := \exp(-\exp\{\log(A + x^2)\}^\alpha), \qquad x \in \mathbb{R}, \tag{1.16}$$

$\alpha > 1$, A large enough.

Our main result is:

THEOREM 1.2. *Let $W := e^{-Q} \in SE^*(3)$, and let a_n, $n \geqslant 1$, denote the nth Mhaskar–Rahmanov–Saff number for Q, defined by* (1.5). *Then for $n \geqslant 1$,*

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \sim \frac{n}{a_n} T(a_n)^{1/2} \tag{1.17}$$

and

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \left|1 - \left(\frac{x}{a_n}\right)^2\right|^{1/2} \sim \frac{n}{a_n}. \tag{1.18}$$

Remarks. (a) We note that $T(a_n)$ grows slowly and typically for each $\varepsilon > 0$,

$$T(a_n) = o(\log n)^{1+\varepsilon}, \qquad n \to \infty.$$

For W of (1.15), one has even [8]

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n, \qquad n \to \infty, \tag{1.19}$$

where $\log_j = \log(\log \cdots)$ denotes the jth iterated logarithm.

(b) The proof of (1.17) shows that if $0 < \alpha < \beta < 1$,

$$\lambda_n^{-1}(W^2, a_{tn})\, W^2(a_{tn}) \sim \frac{n}{a_n} T(a_n)^{1/2}, \tag{1.20}$$

uniformly for $t \in [\alpha, \beta]$, $n \to \infty$.

(c) Theorem 1.2 has applications in studying the orthonormal expansions associated with Erdős weights, which the second author hopes to present elsewhere.

(d) Because of the monotonicity properties of Christoffel functions, one can derive analogues of Theorem 1.2 for weights $W_1 \sim W \in SE^*(3)$.

The proofs are presented in Section 2.

2. Proofs

Throughout, $C, C_1, C_2, C_3, \ldots$ denote positive constants independent of n and x; the same symbol does not necessarily denote the same constant in different occurrences.

We first gather some results from [9]:

LEMMA 2.1. *Let $W \in SE^*(3)$. Let*

$$\mu_{n,a_n}(x) := \frac{2}{\Pi^2} \int_0^1 \left(\frac{1-x^2}{1-s^2}\right)^{1/2} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)}\, ds, \tag{2.1}$$

$x \in (-1, 1)$, $n \geqslant 1$. *Then there exists $\delta > 0$ such that*

$$\lambda_n^{-1}(W^2, a_n x)\, W^2(a_n x) \frac{a_n}{n} = \mu_{n,a_n}(x) + O(n^{-1/5}), \tag{2.2}$$

uniformly for

$$|x| \leqslant 1 - n^{-\delta}. \tag{2.3}$$

Proof. This is (2.23) of Theorem 2.3 in [9]. ∎

For the reader unfamiliar with μ_{n,a_n}, we note that it is the non-negative density function that solves an integral equation with logarithmic kernel [10, Lemma 5.3, p. 37].

LEMMA 2.2. *Let $W \in SE^*(3)$.*

(a) *Then for $x \in [-1, 1]$, $n \geqslant 1$,*

$$\mu_{n,a_n}(x) \leqslant C_1 T(a_n)^{1/2}. \tag{2.4}$$

(b) *Given $0 < \varepsilon < 1$, there exists n_0 such that*

$$\mu_{n,a_n}(x) \sim 1, \qquad |x| \leqslant 1 - \varepsilon, \qquad n \geqslant n_0. \tag{2.5}$$

(c) *We have for $j = 0, 1, 2$,*

$$a_n^j Q^{(j)}(a_n) \sim n T(a_n)^{j-1/2}, \qquad n \geqslant n_0. \tag{2.6}$$

(d) *For any fixed* $0<\alpha<\beta<\infty$, *as* $n\to\infty$,

$$T(a_{\alpha n})\sim T(a_{\beta n}) \tag{2.7}$$

and

$$1-a_{\alpha n}/a_{\beta n}\sim T(a_n)^{-1}. \tag{2.8}$$

Proof. (a) and (b) are respectively (e) and (c) of Lemma 4.3 in [9], with $R=a_n$.

(c) This is (3.15) of Lemma 3.2(c) in [9].

(d) Firstly, (2.7) is Lemma 3.4(c) in [9]. Also, by Lemma 3.4(b) in [9],

$$a_t'/a_t\sim(tT(a_t))^{-1},\qquad t \text{ large enough.}$$

Hence

$$\begin{aligned}\log\left(\frac{a_{\alpha n}}{a_{\beta n}}\right)&=\int_{\alpha n}^{\beta n} a_t'/a_t\,dt\\&\sim\int_{\alpha n}^{\beta n}(tT(a_t))^{-1}\,dt\\&\sim T(a_n)^{-1}\int_{\alpha n}^{\beta n}t^{-1}\,dt\sim T(a_n)^{-1}.\end{aligned}$$

Then (2.8) follows. ∎

We can now prove a result that may be of independent interest:

THEOREM 2.3. *Let* $W\in SE^*(3)$. *Then given* $0<\alpha<\beta<1$, *for* n *large enough*,

$$\max_{[-1,1]}\mu_{n,a_n}(x)\sim T(a_n)^{1/2} \tag{2.9}$$

$$\sim\min_{t\in[\alpha,\beta]}\mu_{n,a_n}(a_{tn}/a_n). \tag{2.10}$$

Proof. In view of (2.4), it suffices to show that

$$\mu_{n,a_n}(a_{tn}/a_n)\geqslant CT(a_n)^{1/2}, \tag{2.11}$$

uniformly for $t\in[\alpha,\beta]$. First note that $d/du(uQ'(u))=Q'(u)\,T(u)$ is positive in $(0,\infty)$. It is also increasing for large u, since $T(u)$ is increasing in $(0,\infty)$ and $Q'(u)$ is increasing for large u (for $Q''(u)=$

$Q'(u)(T(u)-1)/u>0$, u large). It follows that $uQ'(u)$ is convex for large u. Then for $n \geqslant n_1$, $x \geqslant \frac{1}{2}$ fixed,

$$\Delta_n(s) := \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{a_n s - a_n x} \tag{2.12}$$

is a positive function of $s \in (0, \infty)$ and is also increasing if $s \in (\frac{1}{2}, \infty)$. From (2.1), for $x \in [\frac{1}{2}, 1]$,

$$\begin{aligned} \mu_{n,a_n}(x) &\geqslant C_1 \frac{a_n}{n} (1-x^2)^{1/2} \int_x^1 \Delta_n(s)(1-s^2)^{-1/2}\,ds \\ &\geqslant C_2 \frac{a_n}{n} (1-x)\, \Delta_n(x) \\ &= C_2 \frac{a_n}{n} (1-x) \frac{d}{du} (uQ'(u))|_{u=a_n x} \\ &= C_2 \frac{a_n}{n} (1-x)\, Q'(a_n x)\, T(a_n x). \end{aligned} \tag{2.13}$$

Now if $x = a_{tn}/a_n$, $t \in [\alpha, \beta]$, then by (2.8),

$$1 - x \geqslant 1 - a_{\beta n}/a_n \sim T(a_n)^{-1},$$

while by (2.6) and (2.7),

$$a_n Q'(a_n x)\, T(a_n x) \geqslant a_{\alpha n} Q'(a_{\alpha n})\, T(a_{\alpha n}) \sim n T(a_n)^{3/2}.$$

Then (2.11) follows. ∎

We need one more estimate for μ_{n,a_n}:

THEOREM 2.4. *Let $W \in SE^*(3)$. Then for n large enough,*

$$\max_{[-1,1]} \mu_{n,a_n}(x)(1-x^2)^{1/2} \sim 1. \tag{2.14}$$

Proof. In view of (2.5), it suffices to show that

$$\mu_{n,a_n}(x)(1-x^2)^{1/2} \leqslant C, \qquad \tfrac{3}{4} \leqslant x \leqslant 1. \tag{2.15}$$

Let $\delta(x) := (1-x)/2$. Note that $x + \delta(x) < x + 2\delta(x) = 1$, for $x \in [\frac{3}{4}, 1]$. Let $\Delta_n(s)$ be defined by (2.12). Then from (2.1), for $x \in [\frac{3}{4}, 1]$,

$$\mu_{n,a_n}(x)(1-x^2)^{1/2} \leqslant C_1\,\delta(x)\frac{a_n}{n}\int_0^1 \Delta_n(s)(1-s^2)^{-1/2}\,ds$$

$$= C_1\,\delta(x)\frac{a_n}{n}\left\{\int_0^{x-\delta(x)} + \int_{x-\delta(x)}^{x+\delta(x)} + \int_{x+\delta(x)}^{1}\right\}$$

$$\times\,\Delta_n(s)(1-s^2)^{-1/2}\,ds$$

$$=: C_1\,\delta(x)\frac{a_n}{n}\{I_1+I_2+I_3\}. \tag{2.16}$$

Firstly, since $tQ'(t)$ is positive and increasing in $(0, \infty)$, we see from the definition (2.12) of $\Delta_n(s)$ that

$$I_1 \leqslant \int_0^{x-\delta(x)} \frac{a_n xQ'(a_n x)}{a_n x - a_n s}(1-s^2)^{-1/2}\,ds$$

$$\leqslant C_2 xQ'(a_n x)\,\delta(x)^{-1/2}$$

$$(\text{for } a_n x - a_n s \geqslant a_n\,\delta(x))$$

$$\leqslant C_3 xQ'(a_n x)\,\delta(x)^{-1}\int_x^1 (1-s^2)^{-1/2}\,ds$$

$$(\text{by choice of } \delta(x))$$

$$\leqslant C_4 a_n^{-1}\,\delta(x)^{-1}\int_x^1 a_n sQ'(a_n s)(1-s^2)^{-1/2}\,ds$$

$$\leqslant C_5 a_n^{-1}\,\delta(x)^{-1}\,n, \tag{2.17}$$

by definition (1.5) of a_n. Next, if $s \in [x-\delta(x), x+\delta(x)]$, then for some ξ between x and s,

$$\Delta_n(s) = \frac{d}{du}(uQ'(u))\big|_{u=a_n\xi}$$

$$= T(a_n\xi)\,Q'(a_n\xi)$$

$$\leqslant C_6\,\delta(x)^{-1}\int_{x+\delta(x)}^{x+3\delta(x)/2} T(a_n t)\,Q'(a_n t)\,dt$$

$$(\text{as } T(u)\,Q'(u) \text{ is increasing for large } u)$$

$$\leqslant C_6\,\delta(x)^{-1}\,(x+3\delta(x)/2)\,Q'(a_n(x+3\delta(x)/2))$$

$$\left(\text{as } \int T(u)\,Q'(u)\,du = uQ'(u)\right)$$

$$\leqslant C_7\,\delta(x)^{-3/2}\,a_n^{-1}\int_{x+3\delta(x)/2}^{1} a_n sQ'(a_n s)(1-s^2)^{-1/2}\,ds$$

$$\leqslant C_8\,\delta(x)^{-3/2}\,a_n^{-1}n.$$

Hence

$$I_2 \leqslant C_8\, \delta(x)^{-3/2} \frac{n}{a_n} \int_{x-\delta(x)}^{x+\delta(x)} (1-s^2)^{-1/2}\, ds$$

$$\leqslant C_9\, \delta(x)^{-1} \frac{n}{a_n}. \tag{2.18}$$

Finally,

$$I_3 \leqslant \int_{x+\delta(x)}^{1} \frac{a_n s Q'(a_n s)}{a_n s - a_n x} (1-s^2)^{-1/2}\, ds$$

$$\leqslant a_n^{-1}\, \delta(x)^{-1} \int_{x+\delta(x)}^{1} a_n s Q'(a_n s)(1-s^2)^{-1/2}\, ds$$

$$\leqslant C_{10} a_n^{-1}\, \delta(x)^{-1}\, n. \tag{2.19}$$

Combining (2.16) to (2.19) yields (2.15). ∎

We need one more lemma:

LEMMA 2.5. *Let $W \in SE^*(3)$ and $n \geqslant 1$, $k \geqslant 1$. Then for all polynomials P of degree at most kn,*

$$\|PW^k\|_{L_\infty(\mathbb{R})} = \|PW^k\|_{L_\infty[-a_n, a_n]}. \tag{2.20}$$

Proof. If Q is even and convex, this proof follows from Theorem 2.1 and 2.2 in [12, p. 73]–see [12, p. 77]. In the slightly different case, $W \in SE^*(3)$, the result follows from (5.1) of Theorem 5.1 in [9], with W replaced by W^k. ∎

Proof of (1.17) *of Theorem* 1.2. From Lemma 2.1, for a suitable δ, which remains fixed throughout this proof,

$$\max_{|x| \leqslant a_n(1-n^{-\delta})} \lambda_n^{-1}(W^2, x)\, W^2(x) a_n/n$$

$$= \max_{|x| \leqslant 1-n^{-\delta}} \mu_{n,a_n}(x) + o(1). \tag{2.21}$$

Now from (1.11) and (2.6), for each $\varepsilon > 0$, as $n \to \infty$,

$$a_n Q'(a_n) \sim nT(a_n)^{1/2} = O(nQ'(a_n)^{\varepsilon/2}).$$

We deduce that for each $\eta > 0$, as $n \to \infty$,

$$Q'(a_n) = O(n/a_n)^{1+\eta} = o(n^{1+\eta}).$$

Hence from this last inequality and (1.11), for each $\varepsilon > 0$, as $n \to \infty$,

$$T(a_n) = O(n^{\varepsilon}). \tag{2.22}$$

Then (2.8) shows that if $0 < \alpha < \beta < \infty$ and $0 < \varepsilon < \delta$, for n large enough,

$$a_{\alpha n}/\alpha_{\beta n} \leqslant 1 - n^{-\varepsilon} < 1 - n^{-\delta}. \tag{2.23}$$

Hence Theorem 2.3 shows that

$$\max_{|x| \leqslant 1 - n^{-\delta}} \mu_{n, a_n}(x) \sim T(a_n)^{1/2}. \tag{2.24}$$

So from (2.21),

$$\max_{|x| \leqslant a_n(1 - n^{-\delta})} \lambda_n^{-1}(W^2, x)\, W^2(x) a_n/n \sim T(a_n)^{1/2}. \tag{2.25}$$

Since $\lambda_n^{-1}(W^2, x)$ is a polynomial of degree $\leqslant 2n-2$, Lemma 2.5 yields

$$\begin{aligned}
&\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \frac{a_n}{n} \\
&\quad = \max_{x \in [-a_n, a_n]} \lambda_n^{-1}(W^2, x)\, W^2(x) \frac{a_n}{n} \\
&\quad \leqslant \max_{x \in [-a_n, a_n]} \lambda_{2n}^{-1}(W^2, x)\, W^2(x) \frac{a_n}{n} \\
&\quad = \max_{t \in [-a_n/a_{2n}, a_n/a_{2n}]} \lambda_{2n}^{-1}(W^2, a_{2n} t)\, W^2(a_{2n} t) \frac{a_n}{n} \\
&\quad \leqslant 2 \max_{|t| \leqslant 1 - (2n)^{-\delta}} \lambda_{2n}^{-1}(W^2, a_{2n} t)\, W^2(a_{2n} t) \frac{a_{2n}}{2n} \qquad \text{(by (2.23))} \\
&\quad \sim T(a_{2n})^{1/2} \sim T(a_n)^{1/2},
\end{aligned}$$

by (2.25) and (2.7). ∎

The proof also shows that if $0 < \alpha < \beta < 1$,

$$\lambda_n^{-1}(W^2, a_n x)\, W^2(a_n x) \sim \frac{n}{a_n} T(a_n)^{1/2}, \tag{2.26}$$

uniformly for $x \in [a_{\alpha n}/a_n, a_{\beta n}/a_n]$.

Proof of (1.18) *of Theorem* 1.2. Since $\lambda_n^{-2}(W^2, x)(1 - (x/a_n)^2)$ is a polynomial of degree $\leqslant 4n-2$, Lemma 2.5 shows that

$$\max_{x \in \mathbb{R}} \lambda_n^{-2}(W^2, x)\, |1-(x/a_n)^2|\, W^4(x)$$

$$= \max_{x \in [-a_n, a_n]} \lambda_n^{-2}(W^2, x)\, |1-(x/a_n)^2|\, W^4(x)$$

$$\leqslant \max_{x \in [-a_n, a_n]} \lambda_{2n}^{-2}(W^2, x)\, |1-(x/a_{2n})^2|\, W^4(x)$$

$$= \Big\{ \max_{t \in [-a_n/a_{2n}, a_n/a_{2n}]} \lambda_{2n}^{-1}(W^2, a_{2n}t)\, W^2(a_{2n}t)(1-t^2)^{1/2} \Big\}^2$$

$$\leqslant \Big\{ \max_{|t| \leqslant 1-(2n)^{-\delta}} \lambda_{2n}^{-1}(W^2, a_{2n}t)\, W^2(a_{2n}t)(1-t^2)^{1/2} \Big\}^2$$

(by (2.23), with δ as in Lemma 2.1)

$$\leqslant \left\{ 2\frac{n}{a_n} \max_{|t| \leqslant 1-(2n)^{-\delta}} \mu_{n,a_n}(t)(1-t^2)^{1/2} \right\}^2 \leqslant C_1 \left(\frac{n}{a_n}\right)^2,$$

by Lemma 2.1 and Theorem 2.4, for n large enough. Finally, Lemma 2.1 and Lemma 2.2(b) show that given $0 < \varepsilon < 1$,

$$\lambda_n^{-1}(W^2, x)\, W^2(x) \sim \frac{n}{a_n},$$

uniformly for $|x| \leqslant (1-\varepsilon)a_n$, for n large enough. Hence

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, |1-(x/a_n)^2|^{1/2}\, W^2(x) \geqslant C_2 n/a_n. \quad \blacksquare$$

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